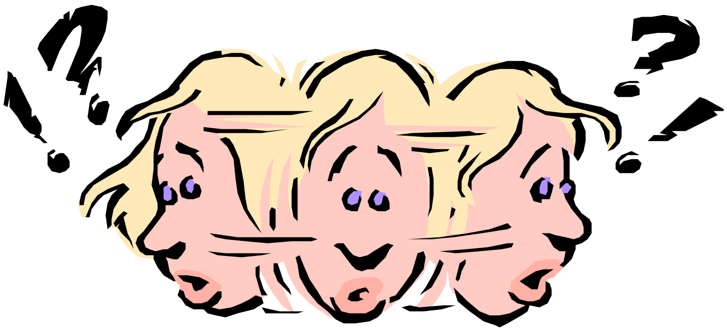
Mental Math



Strategies, routines, ideas, ...and more!

Ideas humbly borrowed
from a variety of
sources

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Some research thoughts to consider…...

National Council of Teachers of Mathematics

The Principles and Standards for School Mathematics put a high priority on both using technology to its full advantage and on learning estimation and mental math skills (National Council of Teachers of Mathematics, 2000).

Mental math and estimation skills. From early elementary school through high school, computing fluently and making reasonable estimates are key standards in mathematics. Given any problem, students need to be able to select appropriate methods of solving: mental calculation, estimation, calculator (computer), or pencil and paper. They also need to develop and use strategies for estimation, so that they will be able to judge the reasonableness of their numerical computations and results (NCTM, 2000).

Use of calculators. The NCTM (2000) states that technology is an indispensable tool for teaching, and ultimately learning and doing mathematics. Students' understanding of abstract mathematical concepts can be greatly enhanced by using the calculator. The effective use of calculators depends greatly on the teacher. Technology should be used as a tool, not as a replacement for teaching. Graphing, visualizing concepts, and computing are enhanced when using the calculator. They can also be used to help teachers bring together for the students the skills of mathematics with the overall understanding of mathematics (NTCM, 2000).

Mental math and estimation skills

Pomerantz (1997) says that mental math, along with pencil and paper and estimation, is essential in the development of mathematical learning skills. It is not only necessary if one does not have a calculator, but, more importantly, it is necessary to check the reasonableness of the calculator answer. In fact, mental math and estimation is even more crucial when using a calculator (Ralston, 1999). Sowder and Kelin (1993, as cited by Reys & Reys, 1998) show that understanding of mathematical concepts increase as students use estimation and mental math – especially when teachers encourage student discovery. An added benefit for the student using estimation and mental math is increased attention span.

The use of mental math and estimation skills. Mental math should be the first choice in problem solving if possible. If not, then estimation of the answer is the second choice. If an exact answer is needed, than a calculator or a standard written computational algorithm (pencil and paper method) would be used, but the estimation of the answer is still crucial for accuracy (Reys & Reys, 1998). If mental math is not encouraged and a student only resorts to a standard written computational algorithm, then the student will see mathematics as only algorithms.

The timetable in teaching mental math and estimation skills. Elementary students should be encouraged to invent their own computational strategies, which involves a great deal of mental math. Estimation should only be used in gaining a sense of numbers for the elementary students such as guessing how many marbles may be in a jar. It is recommended that computational estimation be introduced later in the intermediate grades after the students have a good grasp on large numbers (Reys & Reys, 1998).

As students move into the intermediate grades, they should continue sorting out different strategies as new problems arise, increasing their mental math skills and thus their understanding of mathematical concepts. By the middle grades, the students should have a good grasp of whole numbers and should be extending their conceptual knowledge to include fractions by using mental math of the student’s invention. Estimation should be a major focus at this time. (Reys & Reys, 1998). Rubenstein (2001) says that mental math and estimation should be taught not only in elementary and middle school, but also in the high school and college math courses. Adults use mental math and estimation in their daily lives at home and at work such as interest on loans or investments, shopping, taxes, tips for waiters, traveling, etc. Sharing mental math strategies gives many opportunities to study mathematical properties and to understand them, for example, the inverse operations and the distributive property. Mental math also combats calculator-dependency as students learn calculator-free strategies. Mental math skills give them flexibility as they see the many options before them in problem solving. It gives the students the feeling of empowerment when they are confident in their estimation and mental math skills (Rubenstein, 2001). He believes that every mathematical course has built-in mental math strands that can be used and taught.

Mental math. Ralston (1999) argues that mental math in many instances is less time-consuming than using a calculator and can be very efficient, for both computing an answer and checking an answer (Reys & Reys, 1998). Mental math encourages a student to design his own personal algorithms and thus it promotes a deeper understanding of the concepts (Ralston, 1999). The student who uses mental math well not only improves his number sense, but also he knows how to organize his thought processes which is a useful life skill.

Estimation skills. Estimation skills play a key role in mathematical reflection. Mathematics teachers value reflection of the computed answer, whether the computation was done via calculator or pencil and paper. By estimating an answer, the student can compare his exact results with his estimated results to see if his answer is in a reasonable range (Glasgow, 1998).

Most students only use a few strategies that were taught to them in the classroom to estimate answers, such as rounding numbers off. The best estimators are those that form their own strategies from their understanding of the concept. A teaching method that presents a wide variety of estimation strategies, most of them student-driven based on the understanding of number sense and problem solving strategies, will be much more effective (McIntosh, Reys & Reys, 1992; Sowder, 1992 as cited in Glasgow, 1998).

Many students are not confident in their own estimation skills, especially when it conflicts with a calculator-produced result. The students place more trust in the calculator than in their own estimation skills (Glasgow, 1998). It is imperative that teachers communicate to their students the value of estimation and reflection when using the calculator.

Calculators

There is still much disagreement about the usage of calculators in the classroom. Many purport that calculators have great advantages for students, allowing them to visualize the mathematical concepts without sacrificing the time and energy for tedious computations (Glasgow, 1998; Pomerantz, 1997). Others argue that using calculators will weaken students' ability to perform math or understand its processes (Hunsaker, 1997). To eliminate rote memorization and learning algorithms will only increase mathematical illiteracy. Bracey (1998) argues that to say students need to know how to calculate by rote instead of using a calculator is the same reasoning that Socrates used for oral recitation vs. writing. Socrates argued that learning to write would destroy people's memory. It appears, Bracey (1998) says, that in this country we seem to be of two opinions about the use of the calculator.

The advantages of the use of the calculator. "Calculators are valuable educational tools that allow the students to reach a higher level of mathematical power and understanding" (Pomerantz, 1997, p. 1). When students use calculators, they are able to focus on understanding the concept, setting up the problem and then interpreting the results instead of worrying about tedious calculations (Dick, 1992; Hopkins, 1992 as cited in Beckmann, Senk & Thompson, 1999). Meel (1997, as cited in Bracey, 1998) and Glasgow (1998) agree that by freeing up students from having to expend a lot of time and energy in doing calculations, the use of calculators actually gives the students more time in solving and conceptualizing problems. In addition, Pomerantz (1997) states that the use of calculators allows students more time in developing number sense and mathematical reasoning. Calculators are more efficient, accurate, and faster for laborious computations (Glasgow, 1998; Pomerantz, 1997). Therefore, teachers are able to give students more “real life” problems, even at a younger age that would be otherwise too difficult to grasp without the calculator.

Calculators should not replace mental skills or pencil and paper methods – they should complement them by giving students the ability to solve problems in multiple ways (Pomerantz, 1997). They are also a mathematical equalizer. For students who have always been frustrated with long computations or have given up on math, calculators allow them the ability to experience mathematics and cultivate an understanding of mathematics without being bogged down or hating it. Bracey (1998) believes that one actually has to know more to use a calculator, since the student has to determine whether the answer is reasonable or not. Students have better attitudes and are more confident when they are able to use calculators on assessments (Meel, 1997 as cited by Bracey, 1998). Students who use a calculator on the SAT score slightly higher than those that do not because of less computational error (Lee, 1999).

Use the examples on the following pages to assit you in preparing your students to use mental math strategies.

These questions are from past Grade 8 Common Examinations.

Use the scoring guide information to assist you in discussing strategies for each question.

Direct teaching of mental calculation strategies involves :

- adopting a structured approach so that skills and strategies are developed systematically;
- modeling a strategy using an image, a model, a 'real life' scenario, or some structured apparatus;
- explaining to the children how to do something;
- using a child's error or a less efficient strategy as a starting point for a demonstration of a better strategy;
- encouraging children to compare strategies and improve on their strategies;
- talking about choosing a strategy;
- showing the children how to use a known fact in developing a strategy;
- providing the children with a 'prompt' to help them recall and then use an appropriate number fact.

Children need to be provided with the basic building blocks that enable them to calculate mentally. These include:

- counting, in ones, twos, tens, hundreds, thousands, and any other appropriate unit;

- remembering number facts, and recalling them without hesitation;

- understanding and using relationships between the four operations to find answers and check results;

- developing a repertoire of mental strategies to do calculations, such as counting on or back in tens or ones, bridging through ten, doubling or halving, rounding to perform a calculation, then adjusting the answer.

Difficulties

Look down this list. Can you think of specific children you teach, or have taught, who have some of these difficulties? Consider these problems, drawing upon your own experiences in teaching mathematics.

Have you worked with children or adults, who:

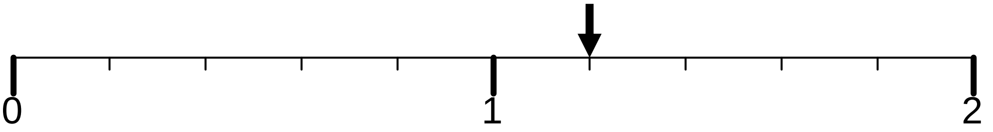
- do almost any addition or subtraction by counting on or back in ones?
- subtract numbers that are close together, such as 42 - 38 by trying to 'take away' or 'count back' 38 from 42, rather than counting up from 38?
- don't recognize that to add 10 or 100 is no more difficult than to add 1?
- don't recognize numbers which are 10, or a multiple of 10, apart?
- never spot a double or a number bond?
- never change a calculation to make it easier, e.g. in 242 - 99 they subtract 99 rather than subtracting 100?
- to calculate mentally, turn to a standard written method and try to visualize it?
- don't see facts which would make the calculation easier, such as the fact that 50 is half of 100 in calculating 36 x 50

If you have students in your room who are currently struggling with these common difficulties, than this is where your teaching of mental math should begin. Once the students are confident with these strategies they will be able to apply their thinking to working with larger numbers

Note: Specific mental mathematics strategies for each question can be found on page 6.

MENTAL MATHEMATICS

1 The arrow on the number line is pointing at which decimal number?



1.2

2 Solve:

5 x 19 = **95**

3 Place the missing number in the following chart.

input	12	3	0	6	9
output	4	1	**0**	2	3

4 Find a value for ☐ which gives you a product that ends with a 6 in the one's place.

54 x **4 or 9** = __ __ 6

5 Solve:

$\frac{1}{2}$ x 16 x 5 = **40**

Circle the **least** value.

3^2 $\sqrt{25}$ [✓ 7.5×10^{-2}]

7 Solve:

$\frac{7}{20}$ = **35** %

8 Insert the correct symbol, **<**, **>**, or **=**, to make the statement true.

2^3 **<** $\sqrt{81}$

MENTAL MATHEMATICS

9 Solve:

$$2\frac{1}{3} \div \frac{1}{3} = \boxed{7}$$

10 Circle the **greater** value.

25% of 70 [✓ 25% of 80]

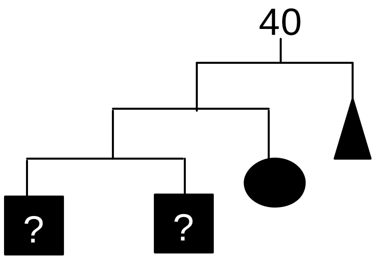
11 Solve:

$$\frac{2\ (9 - 4)\ (5 \times 6)}{5} = \boxed{60}$$

12 Solve:

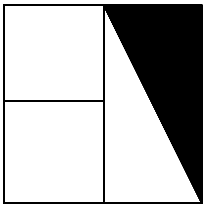
$$4 \div \left(\frac{1}{9}+\frac{2}{9}+\frac{3}{9}+\frac{4}{9}+\frac{5}{9}+\frac{6}{9}+\frac{7}{9}+\frac{8}{9}\right) = 1$$

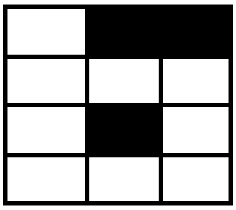
13 What is the value of **one** square, when the mobile is balanced?

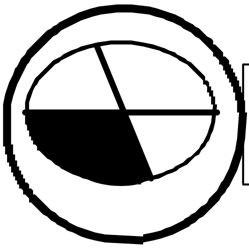


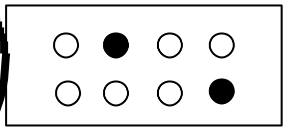
$\boxed{5}$

14 When asked to shade 25% of each model, a student answered the following. (Please circle the **incorrect** model.)









MENTAL MATHEMATICS STRATEGIES

1 Number represented on a number line:
Use the referent numbers and equal partitions to determine the value of intervals

2 Multiply using friendly numbers and compensation:
Round 19 to 20; multiply 5 x 20; compensate by subtracting 5 x 1 from 100

3 Analyze and apply functional pattern:
Use given number pairs to determine the *functional* pattern in a horizontal chart

4 Basic fact and place value knowledge:
Use knowledge that only 4 x 4 and 9 x 4 produce a product with 6 in the ones(units) place

5 Multiply using doubling and halving strategy:
Use knowledge that the product does not change if one factor is halved while another factor is doubled (1/2 x 16 x 5 is the same as 1 x 8 x 5 or 1/2 x 8 x 10)

6 Relative size of number:
Use knowledge of multiplying by a negative exponent

7 Benchmarks in proportional reasoning:
Use knowledge of equivalency of a unit fraction and percent (1/20 =5 %)

8 Relative size of number:
Apply powers and roots to compare values

9 Division involving fractions:
Dividing means saying to oneself " how many groups when *given the size of each group*", or "how many in each group when *given the number of groups*" 2 1/3 ÷ 1/3 is "How many groups of 1/3 are possible in two and one third?"

10 Proportional reasoning:Use knowledge to compare the same percent of two different quantities (eg: 100 % of 70 and 100 % of 80) or uses understanding of percent (eg: 70 % of 25 vs 80 % of 25)

11 Order of operations:
Know and apply the rules when evaluating an expression

12 Compatible numbers:
Recognize that 1/9 and 8/9 =1; 2/9 +7/9=1; and so on; then 4÷(1+1+1+1) is the same as 4÷4

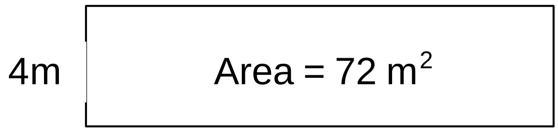
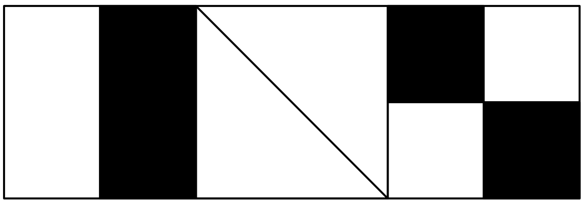
13 Logical reasoning: Use knowledge of a balance scale (all shapes must balance the triangle, so all other shapes must combine to total 20 ; 2 identical squares must balance half of 20; each square must be 5)

14 Relating percent to visual fraction representations: Use knowledge of equivalency, 1/4, 3/12 and 2/8 all represent 25%. The circular model is not partitioned into equivalent pieces.

Note: Specific mental mathematics strategies for each question can be found on page 5.

	MENTAL MATHEMATICS QUESTIONS	ANSWERS
1	$2 \times 16 \times 5 =$ ☐	**160**
2	Circle the fraction that is closest to one-half. [✓ $\frac{7}{12}$] $\frac{4}{2}$ $\frac{1}{3}$	$\frac{7}{12}$
3	☐☐☐☐■ $= \frac{1}{5} = 0.2 =$ ☐ %	**20%**
4	Insert the correct symbol, <, >, or =, to make the statement true. 3^2 ☐ $\sqrt{81}$	**=**
5	$1\frac{1}{5} \div \frac{1}{5} =$ ☐	**6**
6	Estimate the product. $48 \times 614 =$ ☐	**30 000** **or** **30 700**
7	$\frac{4^2 + 3 \times 2}{2} =$ ☐	**11**
8	$6\frac{1}{4} + 6.5 +$ ☐ $= 15$	**2.25 or** **2** $\frac{1}{4}$

<table>
<tr><th></th><th><u>MENTAL MATHEMATICS</u></th><th><u>ANSWERS</u></th></tr>
<tr><td>9</td><td>The arrow on the number line is pointing at what decimal number?

 </td><td>0.6</td></tr>
<tr><td>10</td><td>The rectangles width is 4m, what is its missing length?

 </td><td>8 m</td></tr>
<tr><td>11</td><td>75% of 60 = ☐</td><td>45</td></tr>
<tr><td>12</td><td>Place the missing number in the following chart.
<table><tr><td>input</td><td>3</td><td>8</td><td>2</td><td>6</td><td>7</td></tr><tr><td>output</td><td>21</td><td>56</td><td>14</td><td></td><td>49</td></tr></table></td><td>42</td></tr>
<tr><td>13</td><td>What is the value of the shaded regions?
</td><td>$\frac{1}{3}$</td></tr>
<tr><td>14</td><td>The number 22 is halfway between 15 and what number?
☐</td><td>29</td></tr>
</table>

MENTAL MATHEMATICS STRATEGIES

1

- Looking for compatible numbers:
  The question 2 x 16 x 5 = can be thought of as 2 x 5 x 16 = 10 x 16 = 160

2

- Using benchmarks:
  Students should recognize that $^{7}/_{12}$ is close to $^{6}/_{12}$ or $^{1}/_{2}$.

3

- Understanding visual representations of fractions, decimals and percents:
  Students should be able to move fluently between fractions, decimals and percents and their visual representations.

4

- Understanding relative size of number:
  Students should be able to compute using powers and square roots of whole numbers.

5

- Understanding division:
  Dividing a mixed number dividend by a fraction divisor will result in a quotient that must be larger than the dividend

6

- Using estimation:
  Rounding 48 to 50 and 614 to 600 and then multiplying : 50 x 600 = 30 000; or multiplying 614 by 100 to get 61400 then finding half of the product or 30 700.

7

- Order of operations:
  Understanding the rules used in evaluating expressions

8

- Recognition and conversion of unlike terms to find the missing addend.
  6 $^{1}/_{4}$ can be changed to 6.25: $6.25 + 6.5 + \boxed{} = 15 \rightarrow 12.75 + \boxed{} = 15$; or
  6.5 can be changed to 6 $^{1}/_{2}$ or 6 $^{2}/_{4}$: $6\,^{1}/_{4} + 6\,^{2}/_{4} + \boxed{=}15 \quad 12 \rightarrow ^{3}/_{4} + \quad = \boxed{15}$

9

- Using number lines:
  Understanding the intervals on a number line.

10

- Understanding the relationship between area and an array model.

11

- Commutative property of multiplication; Understanding the relationship between fractions and percents:
  The order of two factors can change and not affect the product; Students should be able to convert 75% to $^{3}/_{4}$, and find $^{3}/_{4}$ of 60.

12

- Recognizing patterns:
  Students should recognize functional patterns in a chart.

13

- Using visual clues in a subdivided region:
  Students should recognize the relationship between each subdivided region and the fractional parts it represents.

14

- Understanding position of number (Number lines):
  Understanding size of intervals when numbers are provided.

ANSWERS

Note: Specific mental mathematics strategies for each question can be found on page 5.

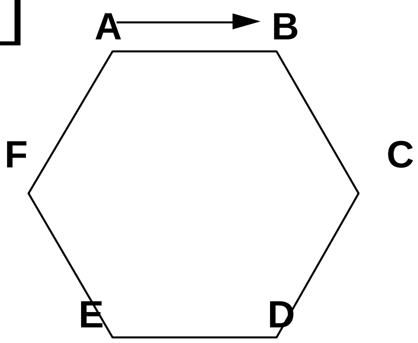
MENTAL MATHEMATICS

300

Q1 4 x 3 x 25 = _____

C

Q2



If you begin at vertex **A** and move in a clock-wise direction, where will you be when you have gone a third of the way around the hexagon?

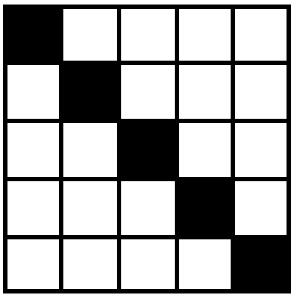
____

100

Q3 25 ÷ $\frac{1}{4}$ = _____

20%

Q4



What percentage of the squares are black?

____

20 x 0.49

Q5 Circle the product which has the greatest value from the three choices below:

20 x 0.44 [✓ **20 x 0.49**] **20 x 0.478**

12
or
ess than 12
(reasonable)

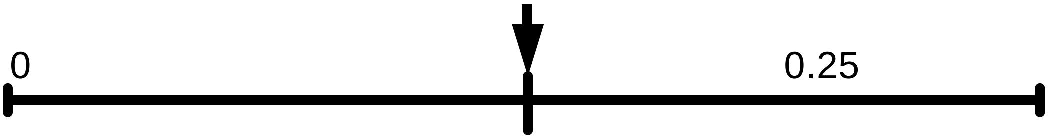
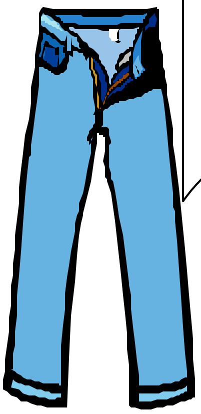
Q6 Estimate the sum of:

4 $\frac{7}{8}$ + 5.87 + 0.978 =

____

ANSWERS

MENTAL MATHEMATICS ...continued

25% of 100	Q7	Circle the greater value: 25% of 95 **or** [✓ 25% of 100]
12	Q8	$\frac{4 \times (4 + 2)}{3} =$ ______
$\frac{3}{4}$	Q9	$\frac{1}{4} + \frac{1}{2} =$ ______
Many possible responses	Q10	List a decimal number that is: **> 2.15** and **< 2.16** ______
32	Q11	64% of 50 = _____
0.125	Q12	 The arrow in the middle of the number line is pointing at what decimal number?________
120	Q13	5 x 24 = ______
$24.00	Q14	 Regular price: $40.00 **Now 40% off!** **No tax** How much will the jeans cost? Circle one of the choices below. **$16.00** [✓ **$24.00**] **$36.00**

MENTAL MATHEMATICS STRATEGIES

Q1
- Looking for compatible numbers:
  The question 4 x 3 x 25 = can be thought of as 4 x 25 = 100 x 3 = 300.

Q2
- Using visual clues in a subdivided region:
  Students should recognize the relationship between the sides of each figure and the fractional parts represented.

Q3
- Understanding division:
  Dividing a whole number dividend by a divisor that is a fraction will result in a quotient that must be larger than the dividend.

Q4
- Connecting pictures with numerical benchmarks:
  The grid with 25 squares can relate to the concept of percent as a grid with 100 squares.

Q5
- Understanding the relative size of decimal numbers:
  The first factor remains the same and 0.49 is the largest second factor.

Q6
- Using estimation:
  Rounding the three numbers up to 5 + 6 + 1 will result in an estimated sum of twelve, so the sum of the original numbers has to be less than 12.

Q7
- Understanding percent/multiplication:
  The 2 proportions expressed show a percentage out of a smaller or a larger region.

Q8
- Order of operations:
  Understanding the rules used to describe the sequence to use in evaluating expressions.

Q9
- Using benchmarks:
  Not relying on a formal algorithm to solve known benchmarks.

Q10
- Place value:
  Aware of tenths, hundredths, and thousandths.

Q11
- Commutative Property of Multiplication:
  The order of 2 factors can change and not affect the product.

Q12
- Number lines.
  Understanding intervals on a number line when the range is provided.

Q13
- Doubling and halving:
  One factor can be doubled if the other factor is halved to create an easier expression. (5 x 24 can become 10 x 12)

Q14
- Using Estimation:
  10% of $40 = $4 so 40% of $40 must be 4 x $4 = $16 off of $40 which is a sale price of $24.